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On the category of S-posets

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0. Introduction. Generalizing usual posets as well as semilattices both of which have been treated from categorical viewpoint in [5] and [8], we study in this article the category of posets acted on by a pomonoid S and the action satisfying the usual properties. Our main results are:

(i) adjunctions from our category to the category of usual posets,

(ii) a structure theorem for projective S-posets, and finally

(iii) if S is a pogroup, our category admits injective hulls.

1. The category of S-posets — SPOS. Let S be a pomonoid which is not necessarily commutative and let E be a poset. We call E a left *S-poset* (the adjective "left" would be omitted in the sequel) if S acts on E in such a way that (i) the action is monotonic in each of the variables, (ii) for $s, t \in S$ and $x \in E$ we have $s(tx)=(st)x$ and (iii) $ex=x$ where e is the identity of S and sx stands for the result of the action of s on x. Let us call such an order on E an *S-order*. A *morphism* from an S-poset E to another S-poset F is a monotonic map which preserves S-action. The class of S-posets and morphisms evidently forms a category, which we denote by SPOS.

2. Congruences in SPOS. An equivalence relation Θ on an S-poset E is called a *congruence* if Θ is compatible with the S-action on E and the quotient set E/Θ can be endowed with an S-order so that the canonical surjection is a morphism in SPOS. Let now Θ be an equivalence on E compatible with S-action and $\mathbf{A}=\{A_1, A_2, \ldots, A_n\}$ be a finite sequence of distinct equivalence classes of Θ. $\mathbf{A}$ is called a *Θ-chain* if each class in $\mathbf{A}$ contains an element which is smaller than some element of the following class. Then Θ is a *congruence iff no element belonging to a member in a Θ-chain is smaller than an element of a previous member in that chain* (cf. [4], p. 177 or [1], p. 42).

If Θ is a congruence on E then the induced S-order on E/Θ is given by $[a]\leqq[b]$ iff there is a Θ-chain from $[a]$ to $[b]$. Moreover, every equivalence relation R on E

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compatible with S-action generates a congruence Θ_R: $a\ \Theta_R b$ iff there is an R-chain from $[a]$ to $[b]$ and another from $[b]$ to $[a]$ (cf. [1], p. 46 or [4], p. 182). Finally if Θ_1 and Θ_2 are congruences on E such that $\Theta_1 \leqq \Theta_2$ then the unique map $E/\Theta_1 \to E/\Theta_2$ is a morphism.

3. Standard constructions in SPOS. Let E be an S-poset. The usual notions of *S-subposet, S-subposet generated by a subset X of E and convex S-subposet,* etc., can be defined in the obvious way. The convex S-subposet generated by X will be denoted by $\langle X \rangle$.

The usual definitions of monics and epimorphisms carry over to SPOS. However, an epimorphism need not be surjective. Let f: $E \to F$ be a surjective morphism. Then one can check that $\ker f = \Theta$ is a congruence over E and E/Θ, equipped with the smallest order making the natural surjection $E \to E/\Theta$ an order-preserving map, is isomorphic to F. Let $\{E_i\}_{i \in I}$ be a family of S-posets. Then the categorical *product* is the usual cartesian product with product order and the *coproduct* is the disjoint union.

The *equalizer* of f, g: $E \to F$ is j: $G \to E$ where j is the natural injection and G is given by $G = \{x \mid x \in E\colon f(x) = g(x)\}$. The *coequalizer* is $F \to F/\Theta$ where Θ is the congruence generated by the binary relation R over F, where aRb iff there exists an $x \in E$ such that the sets $\{f(x), g(x)\}$ and $\{a, b\}$ are the same in F.

By [7], Theorem 1 and its dual on page 109 we have

3.1. Theorem. SPOS *has arbitrary limits and colimits.*

There is another construction which is peculiar to SPOS. Given a family of S-posets $\{E_i\}_{i \in P}$ indexed by a poset P, the *ordinal sum* $\coprod^\circ_{i \in P} E_i$ of the family is the disjoint union and obvious S-action; the order relation is now given for $x, y \in \coprod^\circ_{i \in P} E_i$ by $x < y$ if $x \in E_i$ and $y \in E_j$ with $i < j$ or else $x \leqq y$ in $E_i = E_j$. The ordinal sum has the universal mapping property (UMP): given a family f_i: $E_i \to F$ of morphisms such that for $x \in E_i$, $y \in E_j$ with $f_i(x) \leqq f_j(y)$ in F, there exists a unique morphism f: $\coprod^\circ_{i \in P} E_i \to F$ with $f \cdot j_i = f_i$ where j_i is the canonical injection of E_i into the ordinal sum.

4. Free S-posets. Let P be a poset. Then a *free S-poset over P* is a pair (E, φ) where E is an S-poset and φ: $P \to E$ is a monotonic map such that for every monotonic map ψ: $P \to F$ into an S-poset F, there is a unique morphism f: $E \to F$ such that $\psi = f \cdot \varphi$.

4.1. Theorem. *Given a poset P there exists a free S-poset E over P and E is unique up to isomorphism.*

Proof. Let $E=\coprod^{\circ}_{i\in P} S_i$ where $S_i=S$ for each i and $\varphi: P\to E$ given by $\varphi(i)=e$, the identity of S in S_i. Then the UMP of the ordinal sum implies the UMP for the pair (E, φ) as given above and the uniqueness is clear.

We shall denote the free S-poset over P by $\mathbf{F}(P)$. It is isomorphic to $P\times S$ where $P\times S$ is the product poset with lexicographic order and S-action only on the second component.

The subset $B=\{b_i=(i, e)\}$ of $P\times S$ has the property: every element of $P\times S$ is a unique multiple of one (and only one) member of B and if $b_i<b_j$ then for any $s, t\in S$, $sb_i<tb_j$. If we call such a family an *ordered base*, then clearly an S-poset E is free over a poset P iff E has an ordered base $\{x_i\}_{i\in P}$ indexed by P. In this case $Sx_i=\langle x_i\rangle$. The poset P is called the *order type* of the free S-poset E. Then two free S-posets E and F are isomorphic in SPOS exactly if their order types are isomorphic in POS — the category of posets.

Not all S-posets are free even if S is a pogroup. For example, let E be the set Z of all integers and S be the full permutation group of E. Then S acquires a poorder from the natural order of Z and the resulting S-poset E is not free.

Let E be an S-poset. Consider the free S-poset over the poset E, $\mathbf{F}(E)=E\times S$ with the map $\varphi: E\to E\times S$ defined by $\varphi(x)=(x, e)$, then there is a unique morphism $\Pi: E\times S\to E$ defined by $\Pi((x, s))=sx$ such that $\Pi\cdot\varphi=I_E$ and we have $\mathbf{F}(E)/\ker\Pi\cong E$. Hence

4.2. Proposition. *Every S-poset is the quotient of a free S-poset.*

Remark. For a systematic study of standard constructions in ordered algebras we refer the reader to [2] and [3].

5. Some functors. An ordinary poset can be considered as an S-poset with trivial S-action. Let POS denote the category of posets and $\mathbf{U}$ be the inclusion functor. In this section, we shall find a left adjoint $\mathbf{H}$ to $\mathbf{U}$ and study the properties of the resulting adjunction.

First observe that a morphism from an S-poset to a poset is just a monotonic map which is constant on each orbit Sx for $x\in E$.

5.1. Proposition. *Let E be an S-poset. Then there is a poset $\mathbf{H}(E)$ and a morphism h_E: $E\to\mathbf{H}(E)$ such that for any morphism f: $E\to X$ into a poset X, there exists a unique monotonic map $\hat{f}$: $\mathbf{H}(E)\to X$ with $\hat{f}\cdot h_E=f$.*

Proof. Let Θ be the congruence on E generated by the binary relation aRb iff there exists $s\in S$ such that $sa=b$. More specifically define $x\Theta y$ for $x, y\in E$ if there exist elements $x=a_0, a_1, \ldots, a_n=y$ such that $Sa_i\cap Sa_{i+1}\neq\emptyset$. Let $\mathbf{H}(E)=E/\Theta$ and h_E be the natural morphism: $E\to E/\Theta$. Suppose f: $E\to X$ is a morphism into a poset X. Then clearly $\ker f=\Theta$ and so there exists a monotonic map $\hat{f}$: $\mathbf{H}(E)\to X$ such that $\hat{f}\cdot h_E=f$.

Now if f: $E\to F$ is a morphism in SPOS, the above construction implies that there exists a unique montonic map f_H: $\mathbf{H}(E)\to\mathbf{H}(F)$ such that $f_H\cdot h_E=h_F\cdot f$ and the correspondence $(\mathbf{H}(\ ),\ (\ )_H)$ defines a functor from SPOS to POS. Let $\eta_E=U\cdot h_E$: $E\to\mathbf{UH}(E)$ be the natural homomorphism. Then we have

5.2. Theorem. **H** *is left adjoint to* **U**.

Proof. The correspondence η: $I_{\mathrm{SPOS}}\to\mathbf{UH}$ is clearly a natural transformation such that η_E: $E\to\mathbf{UH}(E)$ is universal $\to$ from E to U for every E in SPOS. Then the assignment $\varphi f=\mathbf{U}f\cdot\eta_E$: $E\to\mathbf{U}(X)$ for f: $\mathbf{H}(E)\to X$ establishes a bijective correspondence between the respective hom-sets. Now the theorem follows (by [7], Theorem 2, condition (i), p. 81).

Remark. The unit of this adjunction is η and the counit ε: $\mathbf{HU}\to I_{\mathrm{POS}}$ is the natural order isomorphism.

Now let us discuss the associated monad of this adjunction ([7], p. 134). This is given by $\langle\mathbf{UH};\ \eta\colon I_{\mathrm{SPOS}}\to\mathbf{UH};\ \mu\colon\mathbf{UHUH}\to\mathbf{UH}\rangle$ where μ assigns to every object E in SPOS the map $\mathbf{U}\cdot\varepsilon_{\mathbf{H}(E)}$: $\mathbf{UHUH}(E)\to\mathbf{UH}(E)$ given by the rule $[[x]]$ mapped to $[x]$ for each $x\in E$.

If $\langle T,\eta,\mu\rangle$ is a monad in a category X, then an Eilenberg—Moore algebra (in short: EM-algebra) is a pair (x,h) where x is an object (the underlying object of the algebra) and h is an arrow h: $Tx\to x$ of X (called the structure map of the algebra) with the following properties:

(i) hTh: $T^2x\to x$ is the same as $h\cdot\mu_x$ (associative law),

(ii) $h\cdot\eta_x$: $x\to T_x\to x$ is the identity on x ([7], p. 136).

Hence applying this general definition to our situation, we find that an Eilenberg—Moore algebra for the monad above is a pair (E,g) where E is an S-poset and g is a left inverse for h_E such that the associative law above holds.

A morphism f: $(E,g)\to(E',g')$ of Eilenberg—Moore algebras is a morphism in SPOS such that $g'\cdot f_k=f\cdot g$.

Now consider the category of EM-algebras $(\mathrm{SPOS})^T$. This gives rise to an adjunction $\langle\mathbf{H}^T,\mathbf{U}^T,\eta^T,\varepsilon^T\rangle$: $(\mathrm{SPOS})\to(\mathrm{SPOS})^T$ in which $\mathbf{H}^T$ and $\mathbf{U}^T$ are given by the respective assignments

$$\mathbf{U}^T\quad\begin{array}{ccc}(E,g)&\mapsto&E\\ \downarrow f&&\downarrow f\\ (E',g')&\mapsto&E'\end{array}\qquad\text{and}\qquad\mathbf{H}^T\quad\begin{array}{ccc}E&\mapsto&(\mathbf{UH}(E),\mu_E)\\ \downarrow f&&\downarrow \mathbf{UH}(f)\\ E'&\mapsto&(\mathbf{UH}(E'),\mu)\end{array}$$

and $\eta^T=\eta$ and $\varepsilon^T(E,g)=g$ for each algebra in $(\mathrm{SPOS})^T$ (cf. [7], Theorem 1, p. 136). The monad defined by this new adjunction on SPOS is the same as the original monad.

Also, this new adjunction is related to the original adjunction by the comparison functor. This functor $\mathbf{K}$ is from POS to $(\mathrm{SPOS})^T$ with $\mathbf{U}^T\mathbf{K}=\mathbf{U}$ and $\mathbf{KH}=\mathbf{H}^T$. This is defined by $\mathbf{K}(P)=\langle \mathbf{U}(P), \mathbf{U}\varepsilon_P\rangle$ for any P in POS and $\mathbf{K}(f)=\mathbf{U}(f)$: $\langle \mathbf{U}(P), \mathbf{U}\varepsilon_P\rangle \to \langle \mathbf{U}(P), \mathbf{U}\varepsilon_{P'}\rangle$ for any morphism f in POS ([7], Theorem 1, pp. 138, 139). When this functor $\mathbf{K}$ is an isomorphism, the functor $\mathbf{U}$ is called *monadic*. In the present case $\mathbf{U}$ is indeed monadic, and we shall indicate the proof.

A functor $G\colon A\to X$ *creates coequalizers* for a parallel pair $f, g\colon a\to b$ in A when to each coequalizer $u\colon Gb\to z$ of Gf, Gg in X there is a unique object c and a unique arrow $e\colon b\to c$ with $Gc=z$ and $Ge=u$ and when, moreover, this unique arrow is a coequalizer of f and g. Also a fork $a \underset{g}{\overset{f}{\rightrightarrows}} b\to c$ in a category is called an *absolute coequalizer* if it remains a coequalizer under the action of any functor. Hence in particular it is a coequalizer. By Beck's theorem ([7], Theorem 1, p. 147), the functor $\mathbf{U}\colon \mathrm{POS}\to\mathrm{SPOS}$ is monadic iff $\mathbf{U}$ creates coequalizers for those parallel pairs f, g in POS for which $\mathbf{U}f$ and $\mathbf{U}g$ has an absolute coequalizer in SPOS. Now this is easily verified, since a coequalizer is surjective both in POS and SPOS. Hence we have

5.3. Theorem. *The inclusion functor* $\mathbf{U}$ *is monadic.*

On the other hand, let $\mathbf{F}$ be the free S-poset construction. Then it is easily seen that $\mathbf{F}$ defines a functor from POS to SPOS and let $\mathbf{V}$ be the forgetful functor from SPOS to POS. The map $\Phi_P\colon P\to \mathbf{F}(P)$ associated with $\mathbf{F}$ is a natural transformation from I_{POS} to $\mathbf{F}$. Let $\delta_P=\mathbf{V}\Phi_P\colon P\to\mathbf{VF}(P)$. Then δ_P is a universal arrow from P to $\mathbf{V}$. Hence we conclude

5.4. Theorem. $\mathbf{F}$ *is left adjoint to* $\mathbf{V}$.

The unit of this adjunction is δ and the counit is the canonical epimorphism (Prop. 4.2) $\Pi\colon \mathbf{FV}\to I_{\mathrm{SPOS}}$. The associated monad is given by $\langle \mathbf{VF}, S\colon I_{\mathrm{POS}}\to\mathbf{VF}, \sigma\colon \mathbf{VFVF}\to\mathbf{VF}\rangle$ where σ assigns to every object P in POS the map $\mathbf{V}\Pi_{F(P)}$ from $\mathbf{VFVF}(P)\to\mathbf{VF}(P)$ given by the rule $((p, e), e)$ of $\mathbf{VFVF}(P)$ is mapped into (p, e) of $\mathbf{VF}(P)$ for P in POS.

Now an Eilenberg—Moore algebra for the monad above is a pair (P, h) where P is a poset and $h\colon \mathbf{VF}(P)\to P$ is a left inverse for $\delta\colon P\to\mathbf{VF}(P)$. Using the method of ([7], Theorem 1, p. 152) we can show

5.5. Theorem. *The forgetful functor* $\mathbf{V}$ *is monadic.*

Summarising we have

5.6. Theorem. *In* SPOS *the functor* $\mathbf{UV}$ *which trivialises S-action has a left adjoint* $\mathbf{FH}$.

6. Projective S-posets. An S-poset E is called *projective* if every epimorphism to E is a retraction.

By 4.2 an S-poset E is projective iff it is a retract of a free S-poset. The following theorem gives a characterisation of a projective S-poset similar to the one valid for projective modules.

6.1. Theorem. *Let E be an S-poset. Then E is projective iff there exist maps $h: E \to E$, $g: E \to S$ with the following properties:*

(i) *h is a monotonic map which is constant on $\langle x \rangle$ for $x \in E$.*

(ii) *g preserves S-action, and if $x, y \in E$, $x \leqq y$, $h(x) = h(y)$ then $g(x) \leqq g(y)$.*

(iii) *$g(x)h(x) = x$ for every $x \in E$.*

Proof. Suppose E is projective. Then $\Pi: E \times S \to E$ given by $\Pi((x, s)) = sx$ is a retraction, so there exists an S-morphism $f: E \to E \times S$ where $f(x) = (h(x), g(x))$ such that $\Pi((h(x), g(x))) = g(x)h(x) = x$ for every $x \in E$. Thus it remains only to check conditions (i) and (ii) above.

Since f is monotonic h is also monotonic. Also, if $y = tx$ for some $t \in S$ then $f(tx) = (h(tx), g(tx)) = tf(x) = t(h(x), g(x)) = (h(x), tg(x))$. Thus $h(tx) = h(x)$ and $g(tx) = tg(x)$. If, however, $ax \leqq y \leqq bx$ then $f(ax) \leqq f(y) \leqq f(bx)$. Therefore $(h(x), ag(x)) \leqq (h(y), g(y)) \leqq (h(x), bg(x))$. Thus $h(x) = h(y)$ and $ag(x) \leqq g(y) \leqq bg(x)$.

Conversely, given h and g a priori satisfying the above conditions, define $f: E \to E \times S$ by $f(x) = (h(x), g(x))$. Then by (iii) $(\Pi \cdot f)(x) = x$ for every $x \in E$. Also $f(tx) = (h(tx), g(tx)) = (h(x), tg(x)) = t(h(x), g(x)) = tf(x)$ and if $x < y$ then $h(x) < h(y)$ or else $g(x) \leqq g(y)$. Then $(h(x), g(x)) \leqq (h(y), g(y))$ and thus f is an S-morphism and E is a retract of a free S-poset, so it is projective.

Since the map h factors through $h_E: E \to \mathbf{H}(E)$, we have a different, but equivalent formulation of the theorem above.

6.2. Theorem. *Let E be an S-poset. Then E is projective iff there exist maps $h': \mathbf{H}(E) \to E$ and $g: E \to S$ with the following properties:*

(i) *h' is a montonic map.*

(ii) *g preserves S-action and if $x \leqq y$, $h'([x]) = h'([y])$, then $g(x) \leqq g(y)$.*

(iii) *$g(x)h'([x]) = x$ for every $x \in E$, where $[x]$ is the class of x in $H(E)$ for $x \in E$.*

Example. If $E = X \times S$, the free S-poset over X, then $h: X \times S \to X \times S$ is given by $h((x, s)) = (x, e)$ and $g: E \to S$ by $g((x, s)) = s$.

Call an ideal J in S projective if J is a projective S-poset. Then we have

6.3. Theorem. *An S-poset E is projective iff E is isomorphic to an ordinal sum of the form $\coprod\limits_{i \in I}^{\circ} J_i z_i$ where z_i is a suitable element of E and J_i is a projective ideal of S with the property*

(i) *there exists an $s_i \in J_i$ such that $s_i z_i = z_i$, and*

(ii) *$a \leqq b$ in J_i exactly if $az_i \leqq bz_i$.*

Proof. Projective property is stable under isomorphism and ordinal sum, hence sufficiency is clear.

Conversely let E be projective and $h: E \to E$, $g: E \to S$ be the functions given in Theorem 6.1. Then the equivalence classes E_i of h are convex S-subposets. Let $h_i = h|_{E_i}$ and $g_i = g|_{E_i}$. By the previous result E_i is projective with the aid of the maps g_i and h_i; also h_i is constant on E_i and $(g_i h_i)(x) = x$, so g_i is an isomorphism. Thus $g_i(E_i) = J_i$ is a projective ideal of S and if $h_i(E_i) = z_i \in E_i$ then $J_i \cong J_i z_i = E_i$. Consider $E' = \coprod_{i \in I}^{\circ} E_i \cong \coprod_{i \in I}^{\circ} J_i z_i$. Then E' is projective and as sets $E = E'$. However, the identity map $x = g_i(x) z_i$ is a bimorphism, so in particular an epimorphism from E to E' and since E' is projective this is an isomorphism.

6.4. Corollary. *Over a pogroup G, all projective G-posets are free.*

7. Complete S-posets — completion — injectivity. An S-poset E is *complete* if E is a complete lattice and given a family of elements $\{x_i\}$ in E and $s \in S$ we have $s(\bigvee x_i) = \bigvee s x_i$ where $\bigvee$ denotes the supremum. A morphism between complete posets is *complete* if it preserves supremum of arbitrary family of elements.

A *completion* of an S-poset E is a pair (E^*, φ) where E^* is a complete S-poset and $\varphi: E \to E^*$ is a monomorphism with the property that $\varphi(x) < \varphi(y)$ exactly if $x < y$ in E and for any other pair (F, ψ) with the above data there exists a unique complete morphism $f: E^* \to F$ such that $f \cdot \varphi = \psi$.

7.1. Theorem. *Every S-poset E admits a completion, which is unique up to isomorphism.*

Proof. The proof is essentially the same as that of Theorem 2 in [6].

Now call a monomorphism f of an S-poset *strict* if $f(x) < f(y)$ exactly if $x < y$. An S-poset E is *injective* if given a strict monomorphism $g: A \to B$ and a morphism $f: A \to E$, there exists an extension of f to B, $h: B \to E$ such that $h \cdot g = f$.

7.2. Proposition. *An injective S-poset is complete.*

Proof. If E is an injective S-poset and (E^*, φ) its completion then by the definition applied to the identity morphism on E, φ is a coretraction. Hence E is already complete.

For a converse, we have

7.3. Proposition. *Let G be a pogroup. Then a complete G-poset is injective.*

Proof. Let E be a complete G-poset and $g: A \to B$ be a strict monomorphism of S-posets and $f: A \to E$ be a morphism. For $b \in B$ we define $h(b) = \bigvee_{g(a) \leq b} f(a)$. Now $h(b)$ exists in E and clearly h is monotonic; moreover, since g is strict, we have

$h \cdot g = f$. For $s \in G$ we have

$$h(sb) = \bigvee_{g(x) \leqq sb} f(x) \geqq \bigvee_{g(a) \leqq b} f(sa) = s\Big(\bigvee_{g(a) \leqq b} f(a)\Big) = sh(b).$$

Further $sh(b) = sh(s^{-1}(sb)) \geqq ss^{-1}h(sb)$ which gives $sh(b) \geqq h(sb)$. Thus $h(sb) = sh(b)$ and h is an S-morphism.

Noting that a minimal injective extension is a hull, we have

7.4. Corollary. *If G is a pogroup, then the category of G-posets admits injective hulls.*

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